# Modelling and Control of a Hybrid Robotic Arm with Mixed Rigid-Elastic Joints

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***Abstract*—Traditional rigid joints exhibit good accuracy in position control but come at the expense of high interaction forces. Elastic joints, on the other hand, can limit the interaction forces but have poor position control accuracy and slow response times. Thus, combining both rigid and elastic joints in a robot arm can leverage their respective strengths while mitigating their weaknesses. In this paper, a mathematical model for a three degrees of freedom (DOFs) hybrid robotic arm with mixed rigid-elastic joints is introduced in detail. The first two joints are rigid, while the last one is elastic. The last joint is chosen to be elastic because it is the one that directly interacts with the environment, potentially including interactions with humans. This can help limit the interaction forces. Furthermore, an inverse dynamics control approach for this hybrid robotic arm is proposed. Finally, a simulation is carried out to demonstrate the efficiency of the proposed control algorithm.**

## I. Introduction

With the development of technology, robots are coming closer and closer to humans, sharing space and forces. Conventional industrial robots have rigid joints, which helps render high torque and excellent accuracy in position control. However, when interacting with the surroundings and with humans, rigid joints could be harmful due to excessive interaction forces. In contrast, elastic joints present an alternative approach that can mitigate these interaction forces, promoting safety in human-robot interactions. However, this benefit comes at the cost of reduced accuracy in position control and slower response times.

Our research group has contributed to the field of control with a focus on robots incorporating elastic joints. In [1], a two-feedback loops position control algorithm was proposed. The inner loop is a model reference adaptive controller that learns the dynamics of the motors. The outer loop is a fuzzy proportional-integral controller that generates the desired position for the motors. Subsequently, in [2], the proposed algorithm was tested on a real mechanical model of a two degrees of freedom (DOFs) robot with elastic actuators. In [3], a novel adaptive sliding mode controller was presented. However, it is worth noting that our investigations in these publications were exclusively centred on robots with all joints being elastic. In practical applications of human-robot interaction (HRI), it is desirable to construct robots with a combination of rigid and elastic joints. Such robots exhibit dynamic characteristics that arise from the interplay between the well-understood dynamics of rigid joints and the nuanced behaviour of elastic joints. Thus, there is a critical need for further research aimed at comprehensively understanding and characterising the dynamics of robots with mixed rigid-elastic joints.

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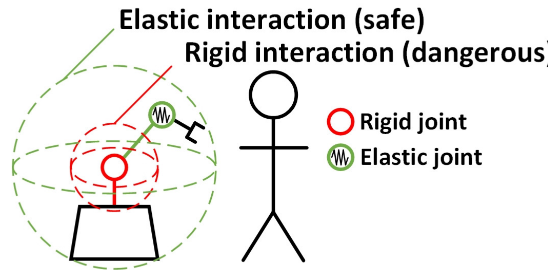


Fig. 1. Human-robot interaction (HRI) with a robot featuring mixed rigid-elastic joints. The green sphere represents the designated safe zone within which the elastic joints engage with the environment. Incorporating this design ensures the safety of both the environment and humans during interactions with the robot. Furthermore, the strategic placement of rigid joints in proximity to the robot's base enhances stability and enables faster responses.

To harness the benefits of both rigid and elastic joints while mitigating their drawbacks, robots equipped with a combination of joint types can be strategically designed. By incorporating rigid joints in their lower sections, these robots can achieve rapid response times and exceptional position accuracy. Additionally, they can exhibit robustness in the face of external environmental forces, such as gravity and collisions. In contrast, the upper sections of these robotic arms may feature joints designed with elasticity, enabling them to interact gently with their surroundings. These elastic joints are programmed to deviate from their initial positions when subjected to excessive external forces, facilitating safe and adaptable interactions with humans. They act as a shock absorber. Moreover, the interaction forces can be adjusted by controlling the deviation between the load side and the motor side of these joints. To the best of our knowledge, there is a scarcity of research on hybrid manipulators in the existing literature. This may be attributed to the inherent complexity of analysing the dynamics of such robots, which pose a greater challenge compared to those with entirely rigid or elastic joints. Furthermore, while the rigid joints in the robot can rapidly respond to command signals, they can unintentionally cause oscillation in the elastic joints. This makes controlling hybrid robots more challenging.

One of the first studies on robots with mixed rigid/elastic joints is presented in [4], where dynamic state feedback is proposed. A linear dynamic feedback compensator is introduced for the inputs at the rigid joints to "soften" these joints. However, this approach sacrifices the strength of rigid joints for the stability of the robot manipulator. In [5], virtual decomposition approach [6], [7] is used instead of the Lagrangian model to develop an adaptive control for a mixed rigid/flexible joint manipulator. This helps simplify the control design. However, joint acceleration is still required, which is a practical limitation. On the other hand, a detailed mathematical model as well as a simple and efficient controller for robots with mixed rigid-elastic joints are still missing.

This work focuses on developing a detailed mathematical model for a 3-DOFs hybrid robotic arm. This robotic arm incorporates a mix of rigid and elastic joints. The initial two joints are rigid, while the last one is elastic. This specific choice of elasticity for the last joint is deliberate, as it is the joint primarily responsible for direct interactions with the environment, including potential interactions with humans. This design aims to reduce interaction forces and enhance safety. The underlying idea is illustrated in Figure 1. The presented model not only serves as a foundation for developing effective control algorithms but also facilitates a thorough analysis of the challenging dynamics inherent to this type of robotic system. In the context of mixed rigid-elastic joints, it is crucial to delineate the joint dynamics into two distinct components: the load side and the motor side. The load side encompasses the dynamics governing the angular position of the joint, while the motor side accounts for the angular position dynamics of the associated motor. It is worth noting that the control strategies for rigid and elastic joints diverge significantly. In particular, rigid joints can be controlled directly through the application of motor torques. In contrast, elastic joints necessitate an indirect control approach relying on the manipulation of elastic forces. Within the motor side, the dynamics correspond to the angular position of the motor for elastic joints. By modulating the motor torques on this side, we gain the ability to regulate the angular position deviation between the motors and the joints. This, in turn, enables precise control over both the force and position of the load imposed on elastic joints. To complement this study, we also introduce an inverse dynamics control method tailored for the proposed hybrid robotic arm. In this control algorithm, firstly, the desired control torque is calculated. For rigid joints, these values are directly applied to the motors. On the other hand, these values are utilised to calculate the desired motor position of the elastic joints. Then, the control law for the motors of elastic joints is then designed to track these desired positions. In the control laws, there are terms to compensate for the nonlinear dynamics. Furthermore, there are proportional-derivative terms to track desired positions. Therefore, for the elastic joints, the deviations between the joint and motor angular positions might not become zeros when there are external forces, e.g. gravity. Because the positions of the elastic joints are controlled indirectly by controlling the deviation between the joint and motor angular positions, there is some delay in controlling these joints. In addition, oscillation caused by the elasticity could also make controlling the elastic joints more challenging. Finally, we conduct simulations to showcase the effectiveness of our proposed control algorithm.

This paper is organised as follows. A review of the related research work is given in Section II. In Section III, a mathematical model using the Lagrangian approach is outlined. Successively, an inverse dynamics control algorithm is proposed in Section IV. Simulations and their results are presented in Section V to show the performance of the robot in terms of trajectory tracking. Finally, conclusions are discussed in Section VI.

## II. Literature Review

Much of the research conducted on the application of elastic actuators in robotics primarily focus on robot manipulators equipped exclusively with elastic joints. In [8], a robust controller for flexible joint robot manipulators is proposed. The controller consists of three parts: a model-based computed torque control part, a feedback-based control part, and a $H_\infty$ robust control part. In [9], an adaptive fuzzy global coupled non-singular fast terminal sliding mode control (NFTSMC) is proposed to deal with uncertainties. In [10], an improved sliding mode control strategy based on fuzzy compensation is presented in order to enhance the trajectory tracking accuracy for space flexible robotic arms. A sliding mode controller is designed using $\tanh$ function. Then an adaptive law of fuzzy systems is designed using the Lyapunov stability theory. In [11], an adaptive controller based on backstepping control, singular perturbation decoupling and neural networks is proposed for flexible-joint robot manipulators with bounded torque inputs. In [12], a combination of inverse dynamics control and passivity-based tracking control is introduced for tracking control in the presence of model uncertainties and external disturbances. In [13], a dual-arm elastic joints Newton-Euler algorithm (DA-EJNEA) is proposed for a dual-arm manipulator carrying a common object. In addition, a feedback linearization method based on DA-EJNEA is considered.

While research on elastic actuators in robotics has made strides, there's still a gap to bridge. Most studies focus on either all-elastic or all-rigid robot joints, overlooking the intriguing area where they coexist. We need to explore how to control these mixed joints and handle uncertainties in dynamic environments for better human-robot collaboration.

## III. Mathematical Modelling

In this section, the mathematical model of the robot is derived using Lagrange's method. First, the following assumptions are made:

**A1:** The elasticity region at each joint is limited to where Hooke's law is applicable, and elasticity effects are limited in this area.

**A2:** The rotors of the motors are represented as uniform objects with their centres of mass located on the axis of rotation.

The robotic arm is an open kinematic chain consisting of $N$ links where $N_e$ out of $N$ joints are elastic and the rest $N_r = N - N_e$ joints are rigid. Let $q = (q_1 \ q_2 \ ... \ q_N)^T$, $\theta = (\theta_1 \ \theta_2 \ ... \ \theta_{N_e})^T$, and $q_e$ denote the vector of link angular positions, the vector of motor angular positions of the elastic joints, and the vector collecting only components of the elastic joints from $q$, respectively.

The kinetic energy of the system involves the kinetic energy of the centre mass of the links and motors:

$$T(q, \dot{q}, \dot{\theta}) = T_{link}(q, \dot{q}) + T_{motor}(q, \dot{q}, \dot{\theta})$$

The kinetic energy of the links results from the combined contributions of translational and rotational movements of their respective centers of mass:

$$\begin{aligned} T_{link}(q, \dot{q}) &= \frac{1}{2}\sum_{i=1}^{N}\left(m_i {v_{cli}}^T v_{cli} + {\omega_i}^T I_i \omega_i\right) \\ &= \frac{1}{2}\dot{q}^T\left[\sum_{i=1}^{N}\left(m_i {J_{vli}}^T J_{vli} + {J_{\omega i}}^T I_i J_{\omega i}\right)\right]\dot{q} \\ &= \frac{1}{2}\dot{q}^T M_l(q)\dot{q} \end{aligned} \quad (1)$$

where $m_i$ is the mass of link $i$ without its motors, $I_i$ is the inertia tensor of link $i$, expressed in frame $i+1$, $v_{cli}$ and $\omega_i$ are respectively the translational and rotational velocities of link $i$, $J_{vli}$ and $J_{\omega i}$ are respectively the Jacobian matrices of the translational and rotational velocities, and

$$M_l(q) = \sum_{i=1}^{N}\left(m_i {J_{vli}}^T J_{vli} + {J_{\omega i}}^T I_{li} J_{\omega i}\right)$$

With each motor $i$, a frame $M_i$ is defined as attaching to rotor $i$ and its z-axis along the motor's rotation axis. The kinetic energy of the motors is:

$$T_{motor}(q, \dot{q}, \dot{\theta}) = \frac{1}{2}\sum_{i=1}^{N}\left(\rho_i {v_{cmi}}^T v_{cmi} + {\omega_{mi}}^T I_{mi}\omega_{mi}\right) \quad (2)$$

where $\rho_i$ is the mass of motor $i$, $I_{mi}$ is the inertia tensor of motor $i$, expressed in frame $M_i$, and $v_{cmi}$ and $\omega_{mi}$ are respectively the translational and rotational velocities of motor $i$. Introduce the Jacobian matrices for the translational movement of the motors:

$$v_{cmi} = J_{vmi}\dot{q}$$

The Jacobian matrices for the rotational movement are calculated according to the type of the joints. If joint $i$ is the k-th elastic joint, then:

$$\omega_{mi} = J_{\omega mi}\dot{q} + z_0\eta_i\dot{\theta}_k \quad (3)$$

where $z_0 = [0 \ 0 \ 1]^T$ is the rotation axis of motor $i$ represented in frame $M_i$, and $\eta_i$ is the transmission ratio of the gearbox. If joint $i$ is a rigid joint, then:

$$\omega_{mi} = J_{\omega mi}\dot{q} \quad (4)$$

Let $\sum_r$ and $\sum_s$ define the sum including all rigid and all elastic joints respectively. Then, by substituting (3) and (4) into (2), we obtain:

$$\begin{aligned} T_{motor}(q, \dot{q}, \dot{\theta}) =& \frac{1}{2}\sum_{i=1}^{N}\rho_i\dot{q}^T {J_{vmi}}^T J_{vmi}\dot{q} \\ &+ \frac{1}{2}\sum_{r}\dot{q}^T {J_{\omega mi}}^T I_{mi} J_{\omega mi}\dot{q} \\ &+ \frac{1}{2}\sum_{s}\left(\dot{q}^T {J_{\omega mi}}^T + {z_0}^T\eta_i\dot{\theta}_k\right) I_{mi}\left(J_{\omega mi}\dot{q} + z_0\eta_i\dot{\theta}_k\right) \\ =& \frac{1}{2}\dot{q}^T\left[\sum_{i=1}^{N}\left(\rho_i {J_{vmi}}^T J_{vmi} + {J_{\omega mi}}^T I_{mi} J_{\omega mi}\right)\right]\dot{q} \\ &+ \frac{1}{2}\sum_{s} I_{mzi}\eta_i^2\dot{\theta}_k^2 + \dot{q}^T\sum_{s}\left({J_{\omega mi}}^T z_0 I_{mzi}\eta_i\dot{\theta}_k\right) \\ =& \frac{1}{2}\dot{q}^T M_m(q)\dot{q} + \frac{1}{2}\dot{\theta}^T B\dot{\theta} + \dot{q}^T S(q)\dot{\theta} \end{aligned} \quad (5)$$

where $M_m(q)$ is the $N \times N$ mass matrix, $B = diag(B_1, B_2, ..., B_N)$ is the $N_e \times N_e$ diagonal inertia matrix, $S(q) = [S_1(q) \quad S_2(q) \quad ...S_N(q)]$ is the $N \times N_e$ coupling matrix with $S_i(q)$ denoting the i-th column of $S(q)$, $I_{mzi}$ is the constant scalar moment of inertia of the rotor about the rotation axis in the matrix $I_{mi}$, and:

$$\begin{aligned} M_m(q) &= \sum_{i=1}^{N}\left(\rho_i {J_{vmi}}^T J_{vmi} + {J_{\omega mi}}^T I_{mi} J_{\omega mi}\right) \\ B_i &= \eta_i^2 I_{mzi} \\ S_i(q) &= \eta_i I_{mzi} {J_{\omega mi}}^T z_0 \end{aligned}$$

Then, by summing (1) and (5)

$$T(q, \dot{q}, \dot{\theta}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}\dot{\theta}^T B\dot{\theta} + \dot{q}^T S(q)\dot{\theta}$$

where

$$M(q) = M_l(q) + M_m(q)$$

The potential energy is a combined contribution originating from both gravity and the elasticity of the joints. The potential energy due to the gravity is calculated as:

$$U_g(q) = -\sum_{i=1}^{N}\rho_i g P_{mi}^0 - \sum_{i=1}^{N} m_i g P_{li}^0$$

where $g = [0 \ 0 \ -9.81]^T$ is the gravitational acceleration vector, $P_{mi}^0$ and $P_{li}^0$ are respectively the positions of the centre of mass of the motor and the link. The potential energy due to the joint elasticity is calculated as:

$$U_e(q_e, \theta) = \frac{1}{2}(q_e - \theta)^T K(q_e - \theta)$$

The Lagrangian of the system is:

$$L = T(q, \dot{q}, \dot{\theta}) - U_g(q) - U_e(q_e, \theta)$$

Euler-Lagrange's equations are written as follows:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}}\right) - \dfrac{\partial L}{\partial q} = \tau_{mr} \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}}\right) - \dfrac{\partial L}{\partial \theta} = \tau_{me} \end{cases}$$

$$\Leftrightarrow \begin{cases} M\ddot{q} + S\ddot{\theta} + C\dot{q} + C_1\dot{\theta} + G(q) = \tau_{mr} - U_{eq} = \tau_q \\ S^T\ddot{q} + B\ddot{\theta} + C_2\dot{q} + U_{e\theta} = \tau_{me} \end{cases} \quad (6)$$

where $C$, $C_1$, $C_2$ are the Coriolis and centrifugal terms, $G$ is the gravitational term, $\tau_{mr}$ and $\tau_{me}$ are respectively the motor torque of the rigid and elastic joints, and

$$\tau_{mr} = [\tau_{mr1}\ \tau_{mr2}\ \dots\ \tau_{mrN}]^T$$

$$\tau_{mri} = \begin{bmatrix} 0 & \text{if the joint } i \text{ is an elastic joint} \\ \tau_{mri}(t) & \text{if the joint } i \text{ is a rigid joint} \end{bmatrix}$$

$$U_{eq} = \frac{\partial U_e(q_e, \theta)}{\partial q} = (U_{eq1}\ U_{eq2}\ \dots\ U_{eqN})^T,$$

$$U_{eqi} = \begin{bmatrix} K_i(q_i - \theta_k) & \text{if joint } i \text{ is the k-th elastic joint} \\ 0 & \text{if joint } i \text{ is a rigid joint} \end{bmatrix}$$

$$\tau_q = [\tau_{q1}\ \tau_{q2}\ \dots\ \tau_{qN}]^T$$

$$\tau_{qi} = \begin{bmatrix} K_i(q_i - \theta_k) & \text{if joint } i \text{ is the k-th elastic joint} \\ \tau_{mri}(t) & \text{if joint } i \text{ is a rigid joint} \end{bmatrix}$$

$$U_{e\theta} = \frac{\partial U_e(q_e, \theta)}{\partial \theta} = K(\theta - q_e)$$

## IV. Inverse Dynamics Control

In this section, an inverse dynamics control algorithm for robots with mixed rigid-elastic joints is proposed. Let $q_d$ denote the desired load position vector. Then, the error between the desired load position and the actual load position is

$$e_{qd} = q_d - q$$

Let $K_{Pq}$ and $K_{Dq}$ denote positive diagonal matrices equivalent to proportional and derivative coefficients in a PD controller, respectively. Let's define the reference velocity as:

$$\begin{aligned} \dot{q}_r &= \dot{q}_d + K_{Dq}^{-1}K_{Pq}e_{qd} = \dot{q}_d + \Lambda_q e_{qd} \\ \Lambda_q &= K_{Dq}^{-1}K_{Pq} \end{aligned}$$

The error between the reference velocity and the actual velocity is defined as

$$\sigma_q = \dot{q}_r - \dot{q} = \dot{e}_{qd} + \Lambda_q e_{qd}$$

Now, the control law for the load side is proposed as

$$\tau_{qc} = M(q)\ddot{q}_r + S(q)\ddot{\theta} + C(q, \dot{q})\dot{q}_r + C_1(q, \dot{q})\dot{\theta} + G(q) + K_{Dq}\sigma_q \quad (7)$$

Then, the dynamic equation of the load side in (6) is subtracted from the control law (7), yielding:

$$M(q)\dot{\sigma}_q + C(q, \dot{q})\sigma_q + K_{Dq}\sigma_q = e_{\tau_q} \quad (8)$$

where

$$e_{\tau_q} = \tau_{qc} - \tau_q = [e_{\tau_q 1}\ \dots e_{\tau_q N}]^T$$

$$e_{\tau_q i} = \begin{bmatrix} K_k(\theta_{dk} - \theta_k) & \text{if joint } i \text{ is the k-th elastic joint} \\ 0 & \text{if joint } i \text{ is a rigid joint} \end{bmatrix}$$

and $\theta_{dk}$ is the desired motor angular position.

A Lyapunov function for the load side is introduced as follows:

$$V_1 = \frac{1}{2}\sigma_q^T M(q)\sigma_q + \frac{1}{2}e_{qd}^T 2\Lambda_q^T K_{Dq} e_{qd}$$

By taking its time-derivative, and applying (8), with the skew-symmetry property of the matrix $\dot{M}(q) - 2C(q, \dot{q})$:

$$\begin{aligned} \dot{V}_1 &= \sigma_q^T M(q)\dot{\sigma}_q + e_{qd}^T 2\Lambda_q^T K_{Dq}\dot{e}_{qd} \\ &= -\dot{e}_{qd}^T K_{Dq}\dot{e}_{qd} - e_{qd}^T\Lambda_q^T K_{Dq}\Lambda_q e_{qd} + \sigma_q^T e_{\tau_q} \end{aligned}$$

Next, the desired motor angular positions for the elastic joints can be calculated from (7) as follows:

$$\tau_{qci} = K_k(\theta_{dk} - q_i) \Rightarrow \theta_{dk} = \frac{1}{K_k}\tau_{qci} + q_i$$

Let $\theta_d$ denote the vector representing the desired angular positions of the elastic joint motors, and let $\sigma_{qe}$ represent the vector consisting exclusively of components related to the elastic joints from $\sigma_q$. Subsequently, let's define errors for the motors in a manner analogous to those for the joints:

$$\begin{aligned} e_{\theta d} &= \theta_d - \theta \\ \dot{\theta}_r &= \dot{\theta}_d + K_{D\theta}^{-1}K_{P\theta}e_{\theta d} + \sigma_{qe} = \dot{\theta}_d + \Lambda_\theta e_{\theta d} + \sigma_{qe} \\ \Lambda_\theta &= K_{D\theta}^{-1}K_{P\theta} \\ \sigma_\theta &= \dot{\theta}_r - \dot{\theta} = \dot{e}_{\theta d} + \Lambda_\theta e_{\theta d} + \sigma_{qe} \end{aligned}$$

Notice that if joint $i$ is rigid, its component in the vector $e_{\tau_q}$ is 0. Thus

$$\sigma_q^T e_{\tau_q} = \sigma_{qe}^T K e_{\theta d}$$

Successively, the control law for the motor side is proposed as follows:

$$\tau_{me} = S(q)^T\ddot{q} + B\ddot{\theta}_r + C_2(q, \dot{q})\dot{q} + K(\theta_d - q_e) + K_{D\theta}\sigma_\theta \quad (9)$$

Then, by substituting (9) into (6),

$$B\dot{\sigma}_\theta + K_{D\theta}\sigma_\theta + Ke_{\theta d} = 0 \quad (10)$$

A Lyapunov function including the errors of the motor side is introduced as follows:

$$V_2 = V_1 + \frac{1}{2}\sigma_\theta^T B\sigma_\theta + \frac{1}{2}e_{\theta d}^T K e_{\theta d}$$

By taking its time-derivative and applying (10):

$$\begin{aligned} \dot{V}_2 &= \dot{V}_1 + \sigma_\theta^T B\dot{\sigma}_\theta + e_{\theta d}^T K\dot{e}_{\theta d} \\ &= -\dot{e}_{qd}^T K_{Dq}\dot{e}_{qd} - e_{qd}^T\Lambda_q^T K_{Dq}\Lambda_q e_{qd} + \sigma_q^T e_{\tau_q} \\ &\quad - \sigma_\theta^T(K_{D\theta}\sigma_\theta + Ke_{\theta d}) + e_{\theta d}^T K(\sigma_\theta - \Lambda_\theta e_{\theta d} - \sigma_{qe}) \\ &= -\dot{e}_{qd}^T K_{Dq}\dot{e}_{qd} - e_{qd}^T\Lambda_q^T K_{Dq}\Lambda_q e_{qd} \\ &\quad - \sigma_\theta^T K_{D\theta}\sigma_\theta - e_{\theta d}^T K\Lambda_\theta e_{\theta d} \end{aligned}$$

Since $\dot{V}_2 \prec 0$, the system is stable.

TABLE I
DYNAMIC PARAMETERS

| | link 1 | link 2 |
|---|---|---|
| mass(kg) | 2.5 | 1.5 |
| inertia | $\begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}$ | $\begin{bmatrix} 0.35 & 0.15 & 0.1 \\ 0.15 & 0.4 & 0.2 \\ 0.1 & 0.2 & 0.3 \end{bmatrix}$ |
| | link 3 | motor 1, 2, 3 |
| mass(kg) | 1.0 | 1.0 |
| inertia | $\begin{bmatrix} 0.3 & 0.15 & 0.13 \\ 0.15 & 0.25 & 0.2 \\ 0.13 & 0.2 & 0.25 \end{bmatrix}$ | $\begin{bmatrix} 0.006 & 0 & 0 \\ 0 & 0.006 & 0 \\ 0 & 0 & 0.004 \end{bmatrix}$ |
| | joint 3 | |
| spring stiffness | 100 | |

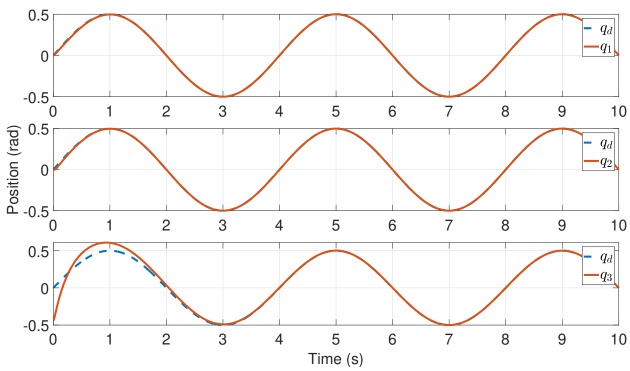


Fig. 2. Load angular positions.

## V. SIMULATION

In this section, the inverse dynamics control is simulated with the dynamic model equations (6). The simulated robot has three DOFs, in which the first two joints are rigid and the last one is elastic. The dynamic parameters are listed in Table I. The reference signal for each motor was a sinusoidal function of the form $0.5\sin(0.5\pi t)$.

The load angular positions are shown in Figure 2. The initial position was $[0\ 0\ -0.4363]^T$ (rad). The actual position followed the desired position $q_d$ with good precision. The position error gradually decreased over time and the root mean square error of joints 1,2, and 3 were respectively 2.57e-07 rad, 2.58e-07 rad, and 1.92e-07 rad.

When controlling elastic joints, the position of the load was controlled indirectly through the position of the motor. Therefore, it was needed that the position of the motor was controlled precisely. The angular position of the motor at the elastic joint is shown in Figure 3.

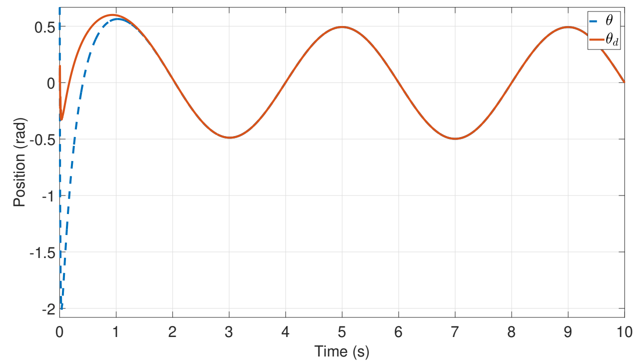


Fig. 3. Motor angular position.

## VI. CONCLUSIONS

In this paper, a mathematical model of robots with mixed rigid-elastic joints was presented. From the mathematical model, the dynamics of this type of robot can be separated into two sides: the load side and the motor side. These sides are interconnected through elastic forces. Subsequently, an inverse dynamics control was proposed. To show the efficiency and stability of the system, a simulation was conducted using Matlab. It is important to note that this controller was designed under the assumption that the dynamic parameters were exactly known. Moving forward, our future research endeavours will focus on the development of more practical control algorithms capable of handling uncertainties. This will involve exploring and implementing adaptive and robust control algorithms to enhance the applicability and reliability of these robotic systems in real-world scenarios. In addition, the robotic structure is going to be tested in more challenging conditions, such as in the presence of interaction, trajectory tracking, etc.

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