Second order systems

Filippo Sanfilippo ¹

¹Department of Engineering Cybernetics, Norwegian University of Science and Technology, 7491 Trondheim, Norway, filippo.sanfilippo@ntnu.no

Trial lecture at the Department of Electrical Engineering and Computer Science, University of Stavanger, Norway, 2017
Introduction

First-order linear systems
Second-order linear systems
Transfer Function
Applications, conclusion and future work

References

About Me

Education:
- PhD in Engineering Cybernetics, Norwegian University of Science and Technology (NTNU), Norway
- MSc in Computer Science Engineering, University of Siena, Italy
- BSc degree in Computer Science Engineering, University of Catania, Italy

Mobility:
- Visiting Fellow, Technical Aspects of Multimodal Systems (TAMS), Department of Mathematics, Informatics and Natural Sciences, University of Hamburg, Hamburg, Germany
- Visiting Student, School of Computing and Intelligent Systems, University of Ulster, Londonderry, United Kingdom
- Granted with an Erasmus+ Staff Mobility for Teaching and Training project

Activities:
- Membership Development Officer for the IEEE Norway Section

Filippo Sanfilippo

Second order systems
Introduction
First-order linear systems
Second-order linear systems
Transfer Function
Applications, conclusion and future work
References

Research topics

Visualisation/Game Development
Augmented Reality/Virtual Reality
Micro-controllers, IoT, Maker Tech.

Mobile Device
Software/Hardware Codesign
Real-time Systems

Artificial Intelligence

Safety-Critical Systems

Filippo Sanfilippo
About Me

**Current position:**
- Filippo Sanfilippo, Postdoctoral Fellow at the Dept. of Eng. Cybernetics, NTNU, Trondheim, Norway

**Current courses:**
- TTK4235 - Embedded Systems (Lecturer)
- Experts in Teamwork - Snake robots (Supervisor)

**Past courses:**
- Real-time Computer Programming (Lecturer)
- Mechatronics, Robots and Deck Machines (Teaching Assistant)
- System Simulation in Matlab/Simulink (Lecturer)

**Current research topic:**
- “SNAKE - Control Strategies for Snake Robot Locomotion in Challenging Outdoor Environments”, project number 240072, supported by the Research Council of Norway through the Young research talents funding scheme

---

**Virtual/real snake robot**
- Gazebo+RViz
- Mamba robot

**Control framework**
- Perception/mapping
- Motion planning
- High-level control
- Position controller
- Velocity controller
- Desired velocity
- Obstacles, pose
- Desired shape/path (position, velocity)
- Motor torques
- Positions, actual velocity
- Contacts, actual shape, actual velocity
- External system commands
- Visual perceptual data
- Tactile perceptual data

**External system commands**
- ROS
- Perception/mapping
- Desired shape/path (position, velocity)
- Motor torques
- Motion planning
- High-level control
- Position controller
- Velocity controller
- Desired velocity
- Obstacles, pose
Canonical homogeneous first-order differential equation:

\[ \tau \frac{dy(t)}{dt} + y(t) = f(t), \quad (1) \]

where \( \tau \neq 0 \) is the system time constant. The characteristic equation is given by:

\[ \tau \lambda + 1 = 0, \quad (2) \]

which has a single root, \( \lambda = -1/\tau \). The system response to an initial condition \( y(0) \) is:

\[ y_h(t) = y(0)e^{\lambda t} = y(0)e^{-t/\tau}, \quad (3) \]
First-order linear systems: s-plane

\[ \lambda_1 = -\frac{1}{\tau} \]

decreasing \( \tau \)
First-order linear systems

\[ \frac{m}{B} \frac{dv_m}{dt} + v_m = \frac{1}{B} F(t); \]
\[ \tau = \frac{m}{B} \]

\[ \frac{B}{K} \frac{dF_K}{dt} + F_K = BV(t); \]
\[ \tau = \frac{B}{K} \]

\[ RC \frac{dv_c}{dt} + v_c = V(t); \]
\[ \tau = RC \]

\[ \frac{L}{R} \frac{di_L}{dt} + i_L = I(t); \]
\[ \tau = \frac{L}{R} \]
The characteristic response of first-order systems

- The canonical homogeneous first-order differential equation is given by:
\[
\tau \frac{dy(t)}{dt} + y(t) = f(t).
\] (4)

The solution (given \(f(t)\) and \(y(0) = 0\)) is the characteristic first-order response.

- The first-order homogeneous solution is of the form of an exponential function:
\[
y_h(t) = e^{-\lambda t}, \quad \lambda = 1/\tau.
\] (5)

- The total response \(y(t)\) is the sum of two components:
\[
y(t) = y_h(t) + y_p(t) = ce^{-t/\tau} + y_p(t),
\] (6)

where \(c\) is a constant to be found from the initial condition \(y(0) = 0\), and \(y_p(t)\) is a particular solution for the given forcing function \(f(t)\).
Second-order linear systems

\[ -F_b - F_k = -b \frac{dx}{dt} - kx = m \frac{d^2x}{dt^2} \Rightarrow m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0. \quad (7) \]

- \( x(0) = x_0, \, v(0) = \dot{x}(0) = v_0 \) are required initial conditions
- assume that \( x(t) \) takes the form \( x(t) = ce^{st} \)

\[ ms^2ce^{st} + bsce^{st} + kce^{st} = 0 \Rightarrow ms^2 + bs + k = 0. \quad (8) \]

\[ s_1 = -\frac{b}{2m} + \frac{\sqrt{b^2 - 4mk}}{2m}, \quad s_2 = -\frac{b}{2m} - \frac{\sqrt{b^2 - 4mk}}{2m}. \quad (9) \]

- \( s_1, s_2 \) are the pole locations (natural frequencies) of the system.
- In most cases \( b2 \neq 4mk \) and the initial condition response will take the form:

\[ x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}. \quad (10) \]

Standard terms

The pole locations are parameterized in terms of the natural frequency $\omega_n$, and the damping ratio $\zeta$ where:

$$
\omega_n = \sqrt{\frac{k}{m}},
$$

(11)

$$
\zeta = \frac{b}{2\sqrt{km}}.
$$

(12)

**Natural frequency and damping ratio:**

- The natural frequency, $\omega_n$, is the frequency at which the system would oscillate if the damping, $b$, were zero.
- The damping ratio, $\zeta$, is the ratio of the actual damping, $b$, to the critical damping, $b_c = 2\sqrt{km}$.

$$
\begin{align*}
\frac{m}{dt} \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx &= 0 \\
\Rightarrow \quad \frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x &= 0.
\end{align*}
$$

(13)

Physically reasonable assumption: the values of $m$, and $k$ are greater than zero (to maintain system order) and that $b$ is non-negative (to keep things stable).
Pole locations

- If $b = 0$, the poles are complex conjugates on the imaginary axis at $s_1 = +j\sqrt{k/m}$, $s_2 = -j\sqrt{k/m}$. This corresponds to $\zeta = 0$, and is referred to as the undamped case.

- If $b^2 - 4mk < 0$ then the poles are complex conjugates lying in the left half of the s-plane. This corresponds to the range $0 < \zeta < 1$, and is referred to as the underdamped case.

- If $b^2 - 4mk = 0$ then the poles coincide on the real axis at $s_1 = s_2 = -b/2m$. This corresponds to $\zeta = 1$, and is referred to as the critically damped case.

- If $b^2 - 4mk > 0$ then the poles are at distinct locations on the real axis in the left half of the s-plane. This corresponds to $\zeta > 1$, and is referred to as the overdamped case.
Second-order linear systems
Standard terms and pole locations
Significant cases

Undamped case ($\zeta = 0$)

The poles lie at $s_1 = j\omega_n$ and $s_2 = -j\omega_n$. 
Underdamped case ($0 < \zeta < 1$)

The poles lie at $s = -\sigma \pm j\omega_d$, where $\sigma = \zeta \omega_n$ is the attenuation, and $\omega_d = \omega_n \sqrt{1 - \zeta^2}$. 
Critically-damped case \((\zeta = 1)\)

The poles lie at \(s_1 = s_2 = -\omega_n\).
Overdamped case ($\zeta = 1$)

The poles lie at $s_1 = -(\zeta - \sqrt{\zeta^2 - 1})\omega_n$ and $s_2 = -(\zeta + \sqrt{\zeta^2 - 1})\omega_n$. 
Laplace transforms and transfer functions

Laplace transforms: method for solving differential equations, converts differential equations in time \( t \) into algebraic equations in complex variable \( s \).

- Defined as the ratio of the Laplace transform of the output signal to that of the input signal (think of it as a gain factor!)
- Contains information about dynamics of a Linear Time Invariant system
Mass-spring-damper system

\[ M \ddot{y}(t) + b \dot{y}(t) + ky(t) = u(t) \]

Assume all initial conditions are zero. Then take Laplace transform,

\[ Ms^2Y(s) + bsY(s) + kY(s) = U(s) \]

\[ \frac{Y(s)}{U(s)} = \frac{1}{Ms^2 + bs + k} \]

**Transfer function**
Mass-Spring-Damper in Simulink

\[ m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0. \]  \hspace{2cm} (14)

Applications


[3]
Learning experience:

- First-order linear systems and characteristic response
- Second-order linear systems and characteristic response
- Several applications
Thank you for your attention

Contact:

- F. Sanfilippo, Department of Engineering Cybernetics, Norwegian University of Science and Technology, 7491 Trondheim, Norway, filippo.sanfilippo@ntnu.no

[4–6]


Introduction

First-order linear systems

Second-order linear systems

Transfer Function

Applications, conclusion and future work

References


